International Journal of Applied Mathematics and Theoretical Physics
2015; 1(3): 19-23
Published online February 19, 2016 (http://www.sciencepublishinggroup.com/j/ijamtp)
doi: 10.11648/j.ijamtp.20150103.11

Extending the 4 × 4 Darbyshire Operator Using *n*-Dimensional Dirac Matrices

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To cite this article:
Paul M. Darbyshire. Extending the 4 × 4 Darbyshire Operator Using *n*-Dimensional Dirac Matrices. *International Journal of Applied Mathematics and Theoretical Physics.* Vol. 1, No. 3, 2015, pp. 19-23. doi: 10.11648/j.ijamtp.20150103.11

Abstract: In this paper, we extend the 4 × 4 Darbyshire operator to develop a new *n*-dimensional formalism using *n*-dimensional Dirac matrices. We then present a set of properties satisfied by the new operator and briefly discuss some areas of interest for potential applications.

Keywords: Darbyshire Operator, Dirac Matrices, Gamma Matrices, Matrix Theory, Quantum Mechanics

1. Introduction

In *non-relativistic* mechanics, the energy, E for a free particle is given by:

$$E = \frac{\mathrm{p}^2}{2m} \tag{1}$$

Where p is the particle three momentum and m is the particle rest mass. Here the *free particle* is in a region of uniform potential set to zero. Introducing quantum mechanical operators for energy, $E \to i\hbar\frac{\partial}{\partial t}$ and momentum, $\mathrm{p} \to -i\hbar\nabla$ leads us to the time-dependent Schrödinger equation for a free particle of the form:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar}{2m}\nabla^2\psi \tag{2}$$

Where i is an imaginary unit, $\hbar$ is the Planck constant divided by 2π, ∇^2 is the Laplacian differential operator, and ψ is the wave function (or *quantum state*) of the particle. The Schrödinger equation suffers from not being relativistically covariant, meaning it does not take into account Einstein's theory of special relativity.

In *relativistic* mechanics, the energy of a free particle is given by:

$$E = \sqrt{c^2\mathrm{p}^2 + m^2c^4} \tag{3}$$

Where c is the speed of light. Inserting the quantum mechanical operators for p and E gives:

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{(-i\hbar\nabla)^2c^2 + m^2c^4}\psi \tag{4}$$

This, however, is a cumbersome expression to work with because the differential operator cannot be evaluated while under the square root sign. To try and overcome this problem, Klein and Gordon instead began with the square of (3), such that:

$$E^2 = c^2\mathrm{p}^2 + m^2c^4 \tag{5}$$

Then:

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2\psi = ((-i\hbar\nabla)^2c^2 + m^2c^4)\psi$$

Which simplifies to:

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi = -\hbar^2c^2\nabla^2\psi + m^2c^4\psi$$

Rearranging terms give us the Klein-Gordon equation of the form:

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla\psi^2 = -\frac{m^2c^2}{\hbar^2}\psi \tag{6}$$

Note that if the particle is at rest (i. e., $m = 0$), (6) simply reduces to the scalar wave equation that describe spin 1 particles. Klein and Gordon proposed that (6) described the quantum mechanical relativistic electron, in fact, it turned out that this was not the case but instead described spin 0 (or *spinless*) particles. In general, particles with half-integer spins (1/2, 3/2, 5/2), are known as *fermions*, while those with integer spins (0, 1, 2), are called *bosons*. A key distinction between the them is that fermions obey the *Pauli exclusion principle*; that is, there cannot be two identical fermions simultaneously having the same quantum numbers. In contrast, bosons obey the rules of *Bose–Einstein statistics* and have no such restriction, and as such can *group together* even if in identical states. No spinless particles have yet been discovered, although the *Higgs boson* is supposed to exist as a spinless particle, according to the Standard Model. Recently, two teams of physicists, working independently, reported preliminary hints of a possible new subatomic particle, which if real, could be either a heavier version of the Higgs boson or a *graviton* [1].

(5) clearly shows that both positive as well as negative energy solutions for each value of p are possible, that is:

$$E = {}^{\pm}\sqrt{c^2\mathrm{p}^2+m^2c^4}$$

For a free particle in a positive energy state, there is no mechanism for it to make a transition to the negative energy state. So, in order to try and explain such ambiguity, Dirac [2] proposed taking only the square root of the positive relativistic energy, which gives:

$$i\hbar\frac{\partial\psi}{\partial t} = c\left[\sqrt{-\frac{\hbar}{2m}\nabla^2+m^2c^2}\right]\psi \quad (7)$$

In order to make sense of (7) Dirac postulated that the square root of a quadratic form in p should be linear in p, such that:

$$\sqrt{c^2\mathrm{p}^2+m^2c^4} = (c\alpha\cdot\mathrm{p}+\beta mc^2) \quad (8)$$

That is:

$$c^2\mathrm{p}^2+m^2c^4 = (c\alpha\cdot\mathrm{p}+\beta mc^2)^2$$

$$= c^2(\alpha\cdot\mathrm{p})^2+\beta^2(mc^2)^2+c\alpha\cdot\mathrm{p}\beta mc^2+\beta mc^2c\alpha\cdot\mathrm{p} \quad (9)$$

It follows that, $(\alpha_i)^2 = \beta^2 = \mathrm{I}$, the *identity matrix* and the following anti-commutator relations are also satisfied:

$$\alpha_i\alpha_j+\alpha_j\alpha_i = \left[\alpha_i,\alpha_j\right]_+ = 2\mathrm{I}\delta_{ij}, i,j = 1,2,3 \quad (10)$$

$$\alpha_j\beta+\beta\alpha_j = \left[\alpha_j,\beta\right]_+ = 0, j = 1,2,3 \quad (11)$$

Where δ_{ij} is the *Kronecker delta* given by:

$$\delta_{ij} = \begin{cases}1, \text{if } i = j\\ 0, \text{if } i \neq j\end{cases} \quad (12)$$

0 is the *null matrix*. So, if α_i and β were simply numbers they would commute and would not satisfy the anti-commutation relations (10) and (11). In order for everything to be consistent, Dirac stated that α_i and β were not numbers but had to be *matrices*. Since these matrices are operators operating on ψ , then ψ itself must be *multicomponent* i. e., a column matrix, at least. From (10) and (11), it is possible to show the α_i and β matrices traceless, and Hermitian, and must have even dimension of at least four. All of this ultimately led to the *Dirac equation* given in its original form as:

$$\left(\beta mc^2+c\left(\sum_{n=1}^{3}\alpha_n p_n\right)\right)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \quad (13)$$

Where $\psi = \psi(x,t)$ is the wave function for the electron of rest mass m with space time coordinates x, t and p_1, p_2, p_3 are the components of the momentum, represented by the momentum operator in the Schrödinger equation. The new elements in (13) are the 4 × 4 matrices α_k and β, and the four-component wave function ψ. There are four components in ψ because evaluation of it at any given point in space is a *bispinor*. It is interpreted as a superposition of a spin-up electron, a spin-down electron, a spin-up positron, and a spin-down positron. The single equation thus unravels into four coupled *partial differential equations* (PDEs) for the wave function ψ of the *spinor* components, ψ_i with $i = 1 \ldots 4$. Moreover, (13) is a relativistic wave equation which describes all spin ½ particles (e. g., *electrons* and *quarks*) for which *parity* is *symmetric*, and is consistent with both the principles of quantum mechanics and the theory of special relativity, and was the first theory to account fully for both of them. In addition, the equation implied the existence of a new form of matter, *antimatter*, as of then unobserved, which was shown to exist experimentally several years later. Moreover, although original thought was that negative energy solutions were not possible, this was not the case. Indeed, Dirac resolved this problem by postulating that all the negative energy states are occupied by fermions (e. g., electrons), and as a result of the Pauli exclusion principle, no two of them could occupy the same quantum state. Thus, a particle with positive energy could be stable since it does not have states left to occupy in the negative region. This, however, meant that in relativistic quantum mechanics there was no such thing as a theory of a single particle. Even a vacuum is filled with infinitely many negative energy particles, and by pumping enough energy into the vacuum matter could be created in the form of electrons excited out of the vacuum, and holes left behind in the sea of negative energy electrons. These holes are positively charged *positrons* with properties the same as that of the electron. In this case, we have to deal with a theory of (infinitely) many particles even if there are only a few in our system to begin with. This ultimately led to the development of *quantum field theory* where electrons and positrons are regarded as being the excitations of the same fundamental matter.

2. Dirac Matrices

As already mentioned, the new elements in (13) are the 4 × 4 *Dirac* (or gamma) matrices α_i and β, and the four-component wave function ψ. Remarkably, the algebraic structure represented by these Dirac (or *gamma*) matrices had been created some 50 years earlier used in the description of *Clifford algebras*. Note also that there are other possible chooses for α_i and β that satisfy the above properties. Two of these, known as the *Weyl* and *Majorana representations*, are also four dimensional and can be convenient for some advanced applications, but these will not be discussed any further here.

The Dirac matrices can be built up from the set of 2 × 2 Pauli spin matrices, σ_1, σ_2 and σ_3, and the 2 × 2 identity matrix, defined by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (14)$$

Such that the Dirac matrices are given as:

$$\beta = \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix},\ \sigma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, i = 1,2,3 \quad (15)$$

A complete set of 4 × 4 Dirac matrices can subsequently be developed as *direct products* (⊗) of the Pauli spin matrices and the identity matrix, such that:

$$\sigma_i = \mathrm{I}_2 \otimes \sigma_i^{(P)}, i = 1,2,3 \quad (16)$$

$$\rho_i = \sigma_i^{(P)} \otimes \mathrm{I}_2, i = 1,2,3 \quad (17)$$

Where $\sigma_i^{(P)}$ are the 2 × 2 Pauli spin matrices. These matrices satisfy the anti-commutation relations, such that:

$$\sigma_i\sigma_j + \sigma_j\sigma_i = [\sigma_i, \sigma_j]_+ = 2\mathrm{I}_4\delta_{ij}, i, j = 1,2,3 \quad (18)$$

$$\rho_i\rho_j + \rho_j\rho_i = [\rho_i, \rho_j]_+ = 2\mathrm{I}_4\delta_{ij}, i, j = 1,2,3 \quad (19)$$

Where $\mathrm{I}_4 = 4 \times 4$ identity matrix. Dirac's original matrices were written α_i, defined by:

$$\alpha_i = \mathrm{E}_{1i} = \rho_i\, \sigma_i, i = 1,2,3 \quad (20)$$

$$\alpha_4 = \mathrm{E}_{30} = \rho_3 \quad (21)$$

Where $\beta \equiv \alpha_4$ can also be used. In this case, we can write:

$$\alpha_1 = \mathrm{E}_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ \alpha_2 = \mathrm{E}_{12} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_3 = \mathrm{E}_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ \alpha_4 = \beta = \mathrm{E}_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (22)$$

It is also possible to premultiply (13) by β and develop a similar set of gamma matrices which satisfy the above anti-commutator properties, such that:

$$\gamma^0 = \beta, \gamma^1 = \beta\alpha_1, \gamma^2 = \beta\alpha_2, \gamma^3 = \beta\alpha_3 \quad (23)$$

And so:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},\ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (24)$$

Or more commonly, as:

$$\gamma^0 = \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, i = 1,2,3 \quad (25)$$

In general, the gamma matrices, $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ are a set of 4 × 4 matrices with specific anti-commutation relations that ensure they generate a matrix representation of a *Clifford algebra*. In addition to the four gamma matrices described above, it is sometimes customary to write a fifth gamma matrix, γ^5 of the form:

$$\gamma^5 = \begin{pmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix} \quad (26)$$

Clifford algebras are a type of associative algebra that can be used to generalise the real numbers, complex numbers, quaternions and several other hyper complex number systems. The theory of Clifford algebras is closely connected to the theory of quadratic forms and orthogonal transformations and have important applications in a number of fields including geometry, theoretical physics and digital image processing.

3. The Darbyshire Operator (Ø)

3.1. The 4 ×4 Construction

In previous works [3-5], a new 4 × 4 matrix operator was developed based on Dirac matrices as defined in (22). This operator, named the *Darbyshire operator* (Ø), was applied to the study of nonlinear coupled wave equations that describe multi-grating formation in a four-wave mixing interaction. Four-wave mixing is a nonlinear process in which light waves interact through the third order electric susceptibility of a material to create phase conjugate waves. The interaction of these waves creates a complex interference pattern, which leads the modulation of the refractive index of the material. This refractive index acts as a thick phase hologram and causes waves to diffract and exchange energy which can result in amplified phase conjugate reflectivity.

Theoretically, the four-wave mixing process often results in a set of nonlinear coupled wave equations in which an analytical solution is generally difficult to achieve, relying frequently on several simplifying assumptions and the use of intensive numerical analysis. To overcome these problems, a novel approach has been based upon the identification of underlying group symmetries within the set of nonlinear coupled wave equations. It was shown in previous works [3-5], that the introduction of the Darbyshire operator led to an analytical solution to the four-wave mixing problem based on group theoretical techniques. Indeed, the application of the operator uncovered a four-dimensional group symmetry as the underlying structure to the nonlinear coupled wave equations. In this paper, we extend the 4 × 4 formalism using *n*-dimensional Dirac matrices to develop an *n*-dimensional version of the Darbyshire operator.

The Darbyshire operator makes use of the symmetry across the trailing diagonal of a 4 × 4 square matrix. If we write the Dirac matrix, E_{11}, such that:

$$E_{11} = \sigma_1 \otimes \sigma_1 = \delta_{i5-j} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \quad (27)$$

Then, the Darbyshire operator, for a 4 × 4 square matrix, A, can be written as:

$$A_{ij}^{\emptyset} = (E_{11})_{ik} A_{kl}^{T} (E_{11})_{lj} = \delta_{i5-k} A_{kl}^{T} \delta_{l5-j} = A_{5-j5-i} \quad (28)$$

Where T is the matrix transpose, thus:

$$A^{\emptyset} = E_{11} A^{T} E_{11} \quad (29)$$

This formulation turned out to be extremely useful in the solution to a set of nonlinear coupled wave equations written in compact matrix form that described four-wave mixing in nonlinear optics [3].

3.2. The Dirac Matrices to n-Dimensions

Pais [6] showed that it was possible to extend the 4 × 4 Dirac matrices to *n*-dimensions. For *n*-dimensions there exists *n* Dirac matrices with dimension $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$, and by defining $\upsilon = \frac{n}{2}$, it is possible to write:

$$\Gamma_1 = \Lambda_\nu$$

$$\Gamma_{2m} = \Lambda_{\nu-m} \otimes \sigma_2^{(\nu-m+1)} \otimes I^{(\nu-m+1)} \otimes \cdots \otimes I^{(\nu)}, m = 1, \ldots \nu - 1$$

$$\Gamma_{2m+1} = \Lambda_{\nu-m} \otimes \sigma_3^{(\nu-m+1)} \otimes I^{(\nu-m+1)} \otimes \cdots \otimes I^{(\nu)}, m = 1, \ldots \nu - 1 \quad (30)$$

$$\Gamma_{2\nu} = \sigma_2^{(1)} \otimes I^{(2)} \otimes \cdots \otimes I^{(\nu)}$$

Where $\Lambda_m = \sigma_1^{(1)} \otimes \sigma_1^{(2)} \otimes \cdots \otimes \sigma_1^{(m)}$ and the superscripts refer to distinct sets of the Pauli matrices. Now, the *n*-dimensional Dirac matrices satisfy the anti-commutator relation given by:

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = [\Gamma_i, \Gamma_j]_+ = 2 I_n \delta_{ij}, i, j = 1 \ldots n \quad (31)$$

Where I_n is the *n*-dimensional identity matrix.

3.3. The n-Dimensional Formalism

Based on (30), it is possible to write the *n*-dimensional Darbyshire operator, in terms of *n*-dimensional Dirac matrices, as:

$$A^{\emptyset} = \Gamma_1 A^{T} \Gamma_1 = \Lambda_\nu A^{T} \Lambda_\nu \quad (32)$$

Where $\Lambda_m = \sigma_1^{(1)} \otimes \sigma_1^{(2)} \otimes \cdots \otimes \sigma_1^{(m)}$. More importantly, for an *n* × *n* square matrix, A the following general properties of the *n*-dimensional formalism can be written down:

(i). $(A^{\emptyset})^{T} = (A^{T})^{\emptyset} = A^{\emptyset}$

(ii). $(A^{\emptyset})^{-1} = (A^{-1})^{\emptyset} = A^{\emptyset}$

(iii). $(A^{\emptyset})^{\dagger} = (A^{\dagger})^{\emptyset}$, where † is the Hermitian conjugate.

(iv). $(A + B)^{\emptyset} = A^{\emptyset} + B^{\emptyset}$

(v). $(AB)^{\emptyset} = A^{\emptyset} B^{\emptyset}$

(vi). $(ABC)^{\emptyset} = A^{\emptyset} B^{\emptyset} C^{\emptyset}$, for cyclic permutations of the matrices. Where $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ are square matrices of the same dimension n.

(vii). $|A^{\emptyset}| = |A|$, where $|\cdot|$ represents the determinant of a matrix.

(viii). $|A^{\emptyset}| = I$, and therefore unitary.

(ix). $\mathrm{Tr}(A^{\emptyset}) = \mathrm{Tr}(A)$, where Tr represents the trace of a matrix.

It is also possible to write the operator as the sum of a *symmetric* and *anti-symmetric* matrix, in the trailing diagonal, such that:

$$A^{\emptyset} = \tfrac{1}{2}(A^{\emptyset} + A^{\emptyset}) + \tfrac{1}{2}(A^{\emptyset} - A^{\emptyset}) \quad (33)$$

Indeed, further analysis of the *n*-dimensional Darbyshire operator is likely to uncover more useful properties, especially when considering symmetry and group theoretical methods.

4. A Brief Review of Potential Applications

We note that higher-dimensional gamma matrices are already used in relativistically invariant wave equations in arbitrary space-time dimensions, notably in *superstring theory*. Superstring theory is based on *supersymmetry* that proposes a type of space time symmetry that relates two basic classes of elementary particles, namely bosons, which have an integer-valued spin, and fermions, which have a half-integer spin. Since the groups generated by these matrices are all the same, it is possible to look for *similarity transformations* that connects them all. This transformation is generated by a respective *charge conjugation* matrix, Indeed, *C-symmetry* refers the symmetry of physical laws under a charge conjugation transformation in which electromagnetism, gravity and the strong interaction all obey, but weak interactions violate. This interestingly relates back to the original analysis by Pais [6], which was used here in our development of the *n*-dimensional Darbyshire operator.

When these higher-dimensional matrices are interpreted as the matrices of the action of a set of orthogonal basis vectors for contravariant vectors in *Minkowski space*, the column vectors on which the matrices act become a space of spinors, on which the Clifford algebra of space time acts. This in turn makes it possible to represent infinitesimal spatial rotations and *Lorentz boosts* (i. e., rotation-free *Lorentz transformations*). Lorentz transformations are coordinate transformations between two coordinate frames that move at constant velocity relative to each other. Historically, the transformations were the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the laws of electromagnetism.

Minkowski space is the mathematical setting in which Einstein's theory of special relativity is most conveniently formulated. In this setting the three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional space-time continuum. While a Euclidean space has only space-like dimensions, a Minkowski space also has one time-like dimension. The symmetry group of a Euclidean space is the Euclidean group and for a Minkowski space it is the *Poincare¨ group*; the Poincare¨ group is the group of isometries of Minkowski space-time. An *isometry* is a way in which the contents of space-time could be shifted that would not affect the proper time along a trajectory between events. If you ignore the effects of gravity, then there are ten basic ways of doing such shifts: translation through time, translation through any of the three dimensions of space, rotation (by a fixed angle) around any of the three spatial axes, or a boost in any of the three spatial directions. These isometries form the Poincare¨ group since there is an identity (no shift, everything stays where it was), and inverses (move everything back to where it was).

5. Conclusions

In this paper, we have extended a new matrix operator that was initially developed using the 4×4 Dirac matrices into its n-dimensional counterpart. We have presented a set of properties satisfied by the new formalism and briefly discussed some areas of interest for potential applications.

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